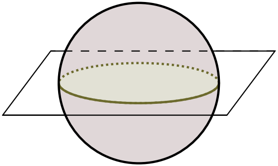
Bourgain's slicing problem and KLS isoperimetry up to polylog



Mira Gordin

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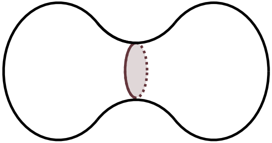
November 1, 2023



after Klartag and Lehec, 2022

Outline

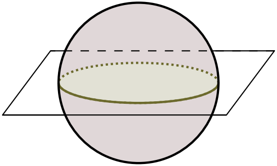
Statement of Bourgain's slicing problem

Bourgain's slicing problem

Let $K \subseteq \mathbb{R}^n$ be a convex body of volume 1. Is there a hyperplane $H \subseteq \mathbb{R}^n$ such that

$$\mathrm{Vol}_{n-1}(K \cap H) > c$$

for some universal constant $c > 0$?



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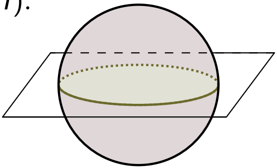
Denote

$$\frac{1}{L_n} := \inf_{K \subseteq \mathbb{R}^n} \sup_{H \subseteq \mathbb{R}^n} \mathrm{Vol}_{n-1}(K \cap H).$$



Statement of Bourgain's slicing problem

Bourgain's slicing problem

Is there a universal constant $C > 0$ such that $L_n \leq C$?

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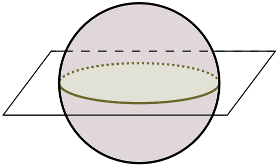


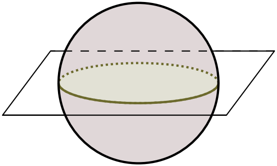
Main Result of Klartag and Lehec, 2022

Theorem 1 (Klartag and Lehec, 2022)

For any $n \geq 2$,

$$L_n \leq C(\log n)^4$$

for some universal constant C.



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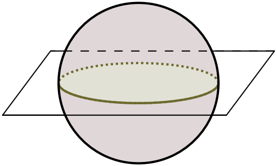

Previous results:

- Chen, 2021 showed $C_\varepsilon n^\varepsilon \geq L_n$ for any $\varepsilon > 0$,
- Bourgain, 1991; Klartag, 2006 showed $C' n^{1/4} \geq L_n$.

Intuition from Busemann-Petty problem

Busemann-Petty problem

Let K, T be centrally symmetric convex bodies $\subseteq \mathbb{R}^n$ satisfying

$$\forall \theta \in S^{n-1}, \quad \mathrm{Vol}_{n-1}(K \cap \theta^{\perp}) \leq \mathrm{Vol}_{n-1}(T \cap \theta^{\perp}). \qquad (*)$$

Is it true that $\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(T)$?

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Answer: it depends!

$$n \leq 4 \quad \rightarrow \quad \text{YES!}$$

$$n \geq 5 \quad \rightarrow \quad \text{NO!}$$

Counterexample for Busemann-Petty problem

$n \geq 10$.

Let $K = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\mathrm{Vol}_n(K) = 1$.

Let $T =$ Euclidean ball of volume $\frac{9}{10}$ centered at origin.

However,

$$\mathrm{Vol}_{n-1}(K \cap \theta^{\perp}) \leq \sqrt{2} < 0.9\sqrt{e} \approx \frac{\left(0.9\Gamma\left(\frac{n}{2}+1\right)\right)^{(n-1)/n}}{\Gamma\left(\frac{n+1}{2}\right)} = \mathrm{Vol}_{n-1}(T \cap \theta^{\perp})$$

K. Ball, 1988.

Equivalent statement to Bourgain's slicing problem

Modified Busemann-Petty problem

Let K, T be centrally symmetric convex bodies $\subseteq \mathbb{R}^n$ satisfying $(*)$.

Is it true that $\mathrm{Vol}_n(K) \leq C\,\mathrm{Vol}_n(T)$ for some universal constant $C > 0$ independent of the dimension n?

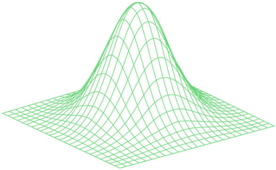
Log-concave & isotropic measures

We call a Borel measure μ on $\mathbb{R}^n$ is *log-concave* if for any compact subsets $A, B \subseteq\in \mathbb{R}^n$ and $0 < \lambda < 1$, we have

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

Classical examples:

- uniform measure on any compact, convex set
- Gaussian measure



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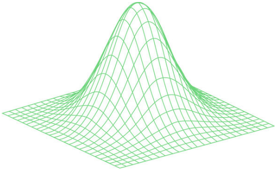


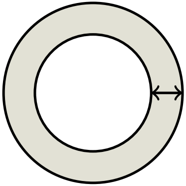
We say a probability measure μ on $\mathbb{R}^n$ with finite second moments is *isotropic* if

$$\int_{\mathbb{R}^n} x_i \, d\mu(x) = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} x_i x_j \, d\mu(x) = \delta_{ij} \quad \text{for } i, j = 1, \ldots, n.$$

Thin-shell constant

The *thin-shell constant* $\sigma_\mu > 0$ of an isotropic, log-concave probability measure μ in $\mathbb{R}^n$ is given by

$$n\sigma_\mu^2 = \mathrm{Var}_\mu \left(|x|^2\right).$$



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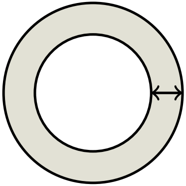

Furthermore, we define the parameter

$$\sigma_n = \sup_\mu \sigma_\mu,$$

where the supremum runs over all isotropic, log-concave probability measure in $\mathbb{R}^n$. Eldan and Klartag showed that

$$L_n \lesssim \sigma_n.$$

Anttila, Ball and Perissinaki 2003

Poincaré constant

The *Poincaré constant* $C_P(\mu)$ of a Borel probability measure μ in $\mathbb{R}^n$ is the smallest constant $C \geq 0$ such that for any locally Lipschitz function $f \in L^2(\mu)$,

$$\mathrm{Var}_\mu(f) \leq C \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu.$$

Poincaré constant upper bound on thin-shell parameter

When μ is an isotropic, log-concave probability measure on $\mathbb{R}^n$,

$$\sigma_n^2 \leq 4C_P(\mu).$$

Proof:

$$n\sigma_\mu^2 = \mathrm{Var}_\mu\left(|x|^2\right)$$

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$$n\sigma_\mu^2 = \mathrm{Var}_\mu\left(|x|^2\right) \leq C_P(\mu) \int_{\mathbb{R}^n} |2x|^2 \, d\mu(x)$$

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Proof:

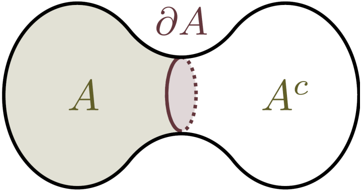
$$n\sigma_\mu^2 = \mathrm{Var}_\mu\left(|x|^2\right) \leq C_P(\mu)\int_{\mathbb{R}^n} |2x|^2\, d\mu(x) = 4n \cdot C_P(\mu)$$

Cheeger constant

Given a probability measure μ in $\mathbb{R}^n$ with log-concave density ρ, its *Cheeger isoperimetric constant* is

$$\frac{1}{\psi_\mu} := \inf_{A \subseteq \mathbb{R}^n} \left\{ \frac{\int_{\partial A} \rho}{\min\{\mu(A), 1-\mu(A)\}} \right\}$$

where the infimum runs over all open sets $A \subseteq \mathbb{R}^n$ with smooth boundary for which $0 < \mu(A) < 1$. Let $\psi_n := \sup_\mu \psi_\mu$.



Connection between Cheeger constant and Poincaré constant

Cheeger's inequality (1970)

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$$\frac{1}{\psi_\mu} \geq \frac{1}{3}\sqrt{\frac{1}{C_P(\mu)}}$$

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When μ is an absolutely-continuous, log-concave probability measure on $\mathbb{R}^n$, we conclude

$$\frac{1}{4} \leq \frac{\psi_\mu^2}{C_P(\mu)} \leq 9.$$

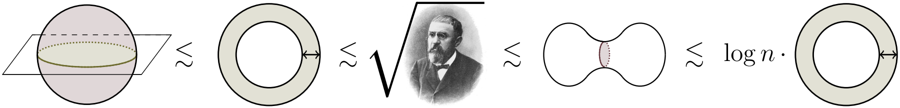
Creating a chain of constants

Summarizing the previous slides, we have that

$$L_n \lesssim \sigma_n \lesssim \sqrt{C_P(\mu)} \lesssim \psi_n \lesssim \log n \cdot \sigma_n,$$

where the supremum runs over all isotropic, log-concave probability measure in $\mathbb{R}^n$.

The last inequality is due to Eldan, 2013.



Kannan-Lovász-Simonovits (KLS) conjecture

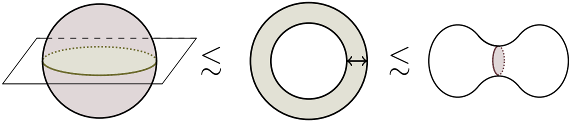
KLS conjecture

There is a universal constant $C > 0$ such that

$$\psi_n \leq C.$$

Clearly, the KLS conjecture implies Bourgain's slicing problem by the chain of inequalities

$$L_n \lesssim \sigma_n \lesssim \psi_n.$$



Main result of Klartag and Lehec, 2022

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For any $n \geq 2$,

$$\psi_n \leq \tilde{C}(\log n)^5$$

for some universal constant $\tilde{C}$.

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Proof ideas: μ isotropic, log-concave probability measure in $\mathbb{R}^n$.

- perform spectral decomposition of the $H^{-1}(\mu)$ norm to obtain estimates on σ_μ.
- use heat flow argument with Eldan's stochastic localization.

Eldan's stochastic localization

Idea: "tilt" our measure μ by some *random* hyperplane θ. Eldan, 2013

Let ρ denote the density of μ. Then let

$$p_{t,\theta}(x) := \frac{1}{Z(t,\theta)} e^{\langle \theta, x \rangle - t|x|^2/2} \rho(x).$$

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Now we consider a stochastic process $(\theta_t)_{t \geq 0}$ that satisfies

$$\theta_0 = 0, \quad d\theta_t = dW_t + a(t,\theta_t)dt, \tag{1}$$

where $(W_t)_{t \geq 0}$ is a standard Brownian motion in $\mathbb{R}^n$.

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Therefore, we can set:

$$p_t(x) := p_{t,\theta_t}(x) \quad a_t = a(t,\theta_t).$$

Stochastic localization continued

Now we obtain the equation for Eldan's stochastic localization

$$p_0(x) = \rho(x), \quad dp_t(x) = p_t(x)\langle x - a_t, dW_t \rangle.$$

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Corollary 2.5 of Klartag and Lehec, 2022

For $t_1 := (C\kappa_n^2 \cdot \log n)^{-1}$, we have for all $t > 0$,

$$\mathbb{E}|a_t|^2 \leq C_1 n \cdot t \cdot \max\left\{1, \frac{t^3}{t_1^3}\right\}$$

for $C, C_1 > 0$ universal constants.

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