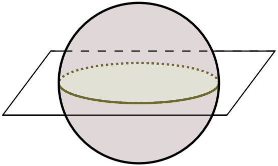
Talk 2: Bourgain's slicing problem and KLS isoperimetry up to polylog



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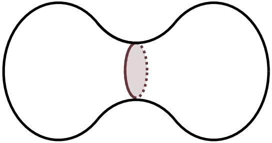
November 3, 2023



after Klartag and Lehec, 2022

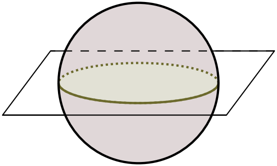
Recall: Bourgain's slicing problem

Bourgain's slicing problem

Let $K \subseteq \mathbb{R}^n$ be a convex body of volume 1. Is there a hyperplane $H \subseteq \mathbb{R}^n$ such that

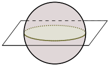
$$\mathrm{Vol}_{n-1}(K \cap H) > c$$

for some universal constant $c > 0$?



Bourgain's slicing problem

Conjecture: $L_n \leq C$



$$\frac{1}{L_n} := \inf_{K \subseteq \mathbb{R}^n} \sup_{H \subseteq \mathbb{R}^n} \mathrm{Vol}_{n-1}(K \cap H)$$

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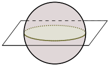

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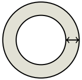
Thin shell parameter



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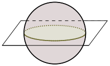

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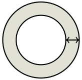

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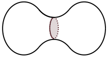
Cheeger constant
KLS conjecture: $\psi_n \leq C$



$$\frac{1}{\psi_\mu} := \inf_{A \subseteq \mathbb{R}^n} \left\{ \frac{\int_{\partial A} \rho}{\min\{\mu(A), 1-\mu(A)\}} \right\}$$

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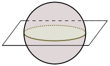

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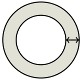

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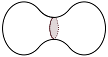

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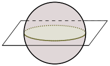

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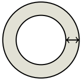

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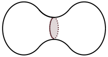

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$$L_n \lesssim \sigma_n \lesssim \psi_n \lesssim \log n \cdot \sigma_n$$

Theorem 1.1 of Klartag and Lehec, 2022

For any $n \geq 2$ and for some universal constant $\tilde{C}$,

$$\psi_n \leq \tilde{C} (\log n)^5.$$

Structure of proof of Theorem 1.1

Proof setting: μ isotropic, log-concave probability measure in $\mathbb{R}^n$.

WLOG we can assume:

- i μ has a smooth positive density ρ,
- ii $\sigma_\mu > \frac{\sigma_n}{2}$ (follows from i).

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Proof ideas:

1. define a heat flow of log-concave measures,
2. obtain functional inequality to estimate σ_μ by upper bounding spectral mass,
3. perform time renormalization to apply Eldan's stochastic localization and optimize over t to obtain final result.

Informal description of Markov semigroup $(P_t)_{t\geq 0}$

For analysts:

$(P_t)_{t\geq 0}$ is a collection of linear operators acting on a suitable function space such that

- $P_0 = \mathrm{Id}$
- $P_t(\mathbb{1}) = \mathbb{1}$
- If $f \geq 0$, $P_t f \geq 0$
- $P_t \circ P_s = P_{t+s}$, $t, s \geq 0$

For probabilists:

For a Markov process $(X_t)_{t\geq 0}$ on a measurable state space E,

$$P_t f(x) = \mathbb{E}[f(X_t) \mid X_0 = x]$$

for $f : E \to \mathbb{R}$.

Bakry, Gentil, Ledoux, et al., 2014

Example: Classical heat semigroup on $\mathbb{R}^n$

Let

$$\gamma_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, \quad t > 0, x \in \mathbb{R}^n$$

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For $f : \mathbb{R}^n \to \mathbb{R}$, we may define

$$P_t f(x) = \int_{\mathbb{R}}^n f(y)\gamma_t(x-y)dy.$$

Log-concave measures along the heat flow

Denote

$$\mu_s := \mu * \gamma_s.$$

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The Laplace operator associated with μ is the operator $L := L_\mu$ given by
$$Lu = \Delta u + \nabla(\log\rho) \cdot \nabla u$$

for a smooth function $u : \mathbb{R}^n \to \mathbb{R}$.

Log-concave measures along the heat flow

By classical results $\mathcal{L}$ is esentially self-adjoint on $L^2(\mu)$ and by the spectral theorem, we may write

$$-L = \int_{-\infty}^{\infty} \lambda \, dE_\lambda$$

for $(E_\lambda)_{\lambda \in \mathbb{R}}$ a certain increasing, right-continuous family of orthogonal projections with

$$\lim_{\lambda \to \infty} E_\lambda = \mathrm{Id} \quad \text{and} \quad \lim_{\lambda \to -\infty} E_\lambda = 0$$

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Denote as ν_f the spectral measure satisfying

$$\nu_f((a, b]) = \langle E_b f, f \rangle - \langle E_a f, f \rangle.$$

Recall the spectral gap of L is

$$\lambda_1 = \frac{1}{C_P(\mu)},$$

i.e. $\nu_f([0, \lambda_1)) = 0$.

Proposition 2.1 of Klartag and Lehec, 2022

Let $f \in L^2(\mu)$ satisfy $\int_{\mathbb{R}^n} f \, d\mu = 0$ and $\|f\|_{L^2(\mu)} = 1$. Then for $s, \lambda > 0$,

$$\langle E_\lambda f, f \rangle_{L^2(\mu)} \leq C(\|Q_s f\|_{L^2(\mu_s)} + s\lambda),$$

for $C > 0$ universal constant.

Brief overview of stochastic calculus

Brownian motion: the unique stochastic process $(W_t)_{t \geq 0}$ satisfying

- $W_0 = 0$, $W_t \sim N(0, t)$
- independent and stationary increments $W_t - W_s$
- a.e. realization of $t \mapsto W_t$ is continuous.

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Ito process:

$$X_t = X_0 + \int_0^t \underbrace{\sigma(s, X_s)}_{\text{noise}} \, dW_s + \int_0^t \underbrace{\mu(s, X_s)}_{\text{drift}} \, ds.$$

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We use the SDE notation

$$dX_t = \underbrace{\sigma(s, X_s)}_{\text{noise}} dW_s + \underbrace{\mu(s, X_s)}_{\text{drift}} ds.$$

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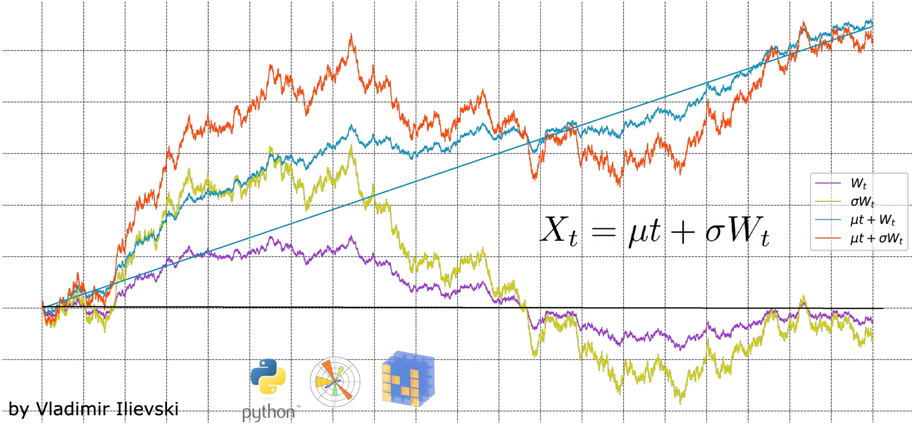

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Ornstein-Uhlenbeck process

$$dX_t = -X_t\, dt + \sqrt{2}\, dW_t$$

If $X_0 \sim N(0, 1)$, then $X_t \sim N(0, 1)$.

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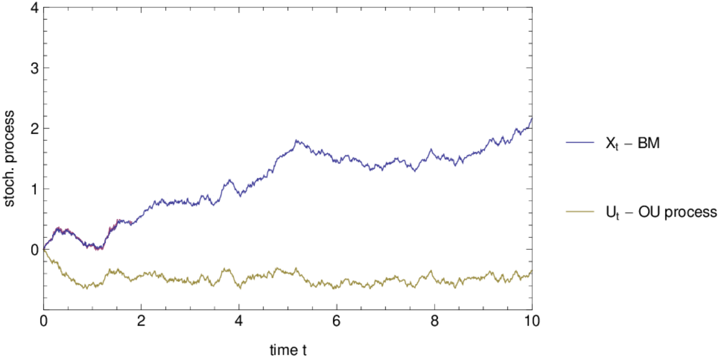

Eldan's stochastic localization

Idea: "tilt" our measure μ by some *random* hyperplane θ. Eldan, 2013

Let ρ denote the density of μ. Then let

$$p_{t,\theta}(x) := \frac{1}{Z(t,\theta)} e^{\langle \theta, x\rangle - t|x|^2/2} \rho(x).$$

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Now we consider a stochastic process $(\theta_t)_{t\geq 0}$ that satisfies

$$\theta_0 = 0, \quad d\theta_t = dW_t + a(t,\theta_t)dt, \tag{1}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}^n$.

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where $(W_t)_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}^n$. By standard arguments, a unique strong solution $(\theta_t)_{t\geq 0}$ to (1) exists.
Therefore, we can set:

$$p_t(x) := p_{t,\theta_t}(x) \quad a_t = a(t,\theta_t).$$

Stochastic localization continued

Now we obtain the equation for Eldan's stochastic localization

$$p_0(x) = \rho(x), \quad dp_t(x) = p_t(x)\langle x - a_t, dW_t\rangle.$$

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Lemma 2.2.

Let $\varphi \in L^1(\mu)$ and $s > 0$. Consider the stochastic process $M_t = \int_{\mathbb{R}^n} \varphi p_t$ defined for $t \geq 0$. Then with $t = 1/s$,

$$\mathbb{E}[M_t^2] = \|Q_s f\|^2_{L^2(\mu_s)}.$$

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Corollary 2.5 of Klartag and Lehec, 2022

For $t_1 := (C\kappa_n^2 \cdot \log n)^{-1}$, we have for all $t > 0$, for $C, C_1 > 0$ universal constants,

$$\mathbb{E}|a_t|^2 \leq C_1 n \cdot t \cdot \max\left\{1, \frac{t^3}{t_1^3}\right\}.$$

H^{-1} norm

We recall that the norm of the Sobolev space $H^1(\mu)$ is given by

$$\|f\|_{H^1(\mu)}^2 = \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu$$

The dual norm for $f \in L^2(\mu)$ with $\int f \, d\mu = 0$ is given by

$$\|f\|_{H^{-1}(\mu)} = \sup \left\{ \int_{\mathbb{R}^n} fu \, d\mu; u \in L^2(\mu) \text{ is locally Lipschitz with } \|u\|_{H^1(\mu)} \leq 1 \right\}$$

We have the following spectral decomposition of the H^{-1} norm

$$\|f\|_{H^{-1}(\mu)} = \int_{\lambda_1}^{\infty} \frac{d\nu_f(\lambda)}{\lambda},$$

here ν_f is the spectral measure of f with respect to L.

Proof of Theorem 1.1

Recall: Proposition 2.1 of Klartag and Lehec, 2022

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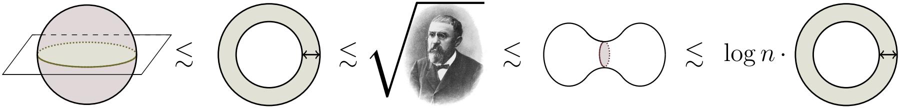
Summary

- $L_n \lesssim \sigma_n \lesssim \psi_n \lesssim \sqrt{\log n}\kappa_n \lesssim \log n \cdot \sigma_n$
- construct a heat flow of measures that allows us to use a variety of analytic and probabilistic techniques to obtain estimates



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