Vector-valued concentration on the symmetric group

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Two perspectives on vector-valued concentration inequalities

Vector-valued analogs of classical concentration inequalities that help us understand high-dimensional random structures

Poincaré inequality

$$\mathrm{Var}(f(X)) \leq C_P \mathbb{E}|\nabla f(X)|^2$$

log-Sobolev inequality

$$\mathrm{Ent}(f^2(X)) \leq C \mathbb{E}|\nabla f(X)|^2$$

Vector-valued functional inequalities that describe phenomena in functional analysis and metric geometry

Example: consequences for metric embeddings of graphs in Banach spaces

$$\downarrow$$

algorithmic applications for embeddings

Pisier's inequalities

Let $(X, \|\cdot\|)$ be a Banach space.

Theorem (Pisier, 1985)

For $f : \mathbb{R}^n \to X$ locally Lipschitz, $G, G' \sim N(0, I_n)$ independent, and $1 \le p < \infty$,

$$\mathbb{E}\,\|f(G) - \mathbb{E}f(G)\|^p \le \left(\frac{\pi}{2}\right)^p \mathbb{E}\left\|\sum_{j=1}^{n} G_j' \frac{\partial f}{\partial x_j}(G)\right\|^p.$$

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Theorem (Pisier, 1985)

For $f : \{-1,1\}^n \to X$, $\varepsilon, \varepsilon' \sim \mathrm{Unif}(\{-1,1\}^n)$ independent,

$$\mathbb{E}\|f(\varepsilon) - \mathbb{E}f(\varepsilon))\|^p \leq C(n)^p \mathbb{E}\left\|\sum_{j=1}^n \varepsilon_j' D_j f(\varepsilon)\right\|^p \tag{1}$$

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- Talagrand (1993) proved sharpness
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Is there another way to think about vector-valued concentration to get the "right" dimensional dependence?

Dimension-free constant on the discrete hypercube

Theorem (Ivanisvili, van Handel, Volberg 2020)

For $f : \{-1,1\}^n \to X$, $\varepsilon \sim \mathrm{Unif}(\{-1,1\}^n)$, and $1 \le p < \infty$

$$\mathbb{E}\|f(\varepsilon) - \mathbb{E}f(\varepsilon)\|^p \le \left(\frac{\pi}{2}\right)^p \int \mathbb{E}\left\|\sum_{j=1}^n \delta_j(t) D_j f(\varepsilon)\right\|^p \mu(dt)$$

where $\mu(dt) := \frac{2}{\pi}\frac{1}{\sqrt{e^{2t}-1}}\,dt$ and $\delta_j(t)$ are appropriately renormalized *biased* Rademacher random variables.

$$D_j f(\varepsilon) := \frac{f(\varepsilon_1, \dots, \varepsilon_j, \dots, \varepsilon_n) - f(\varepsilon_1, \dots, -\varepsilon_j, \dots, \varepsilon_n)}{2}.$$

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We denote by $P_t := e^{t\Delta}$ the standard heat semigroup on G.

Heat semigroup on (G, S)

Let $\{X_t\}$ be a continuous-time random walk on G
with stationary measure $\mu = \mathrm{Unif}(G)$.

heat kernel of the random walk $p_t(x, y) := \mathbb{P}_x(X_t = y)$

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$$\delta_s(t) = \frac{p_t(x, X_t) - p_t(sx, X_t)}{p_t(x, X_t)} = \frac{D_s p_t(x, X_t)}{p_t(x, X_t)}.$$

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$$D_s P_t f(x) = \mathbb{E}_x[f(X_t)\delta_s(t)]$$

Pisier-like inequality on a finite group

Proposition (G., van Handel 2024+)

For any function $f : G \to X$, and $1 \leq p < \infty$, we have

$$\left(\mathbb{E}_\mu \|f - \mathbb{E}_\mu f\|^p\right)^{\frac{1}{p}} \leq \frac{1}{2} \int_0^\infty \left(\mathbb{E}_\mu \left\| \sum_{s \in S} \delta_s(t) D_s f(X_0) \right\|^p \right)^{\frac{1}{p}} dt.$$

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Definition: Rademacher type

We say that Banach space $(X, \|\cdot\|)$ has Rademacher type $q \in [1, 2]$ if there exists a $C \in (0, \infty)$ so that for all $n \geq 1$,

$$\mathbb{E}\left\|\sum_{j=1}^{n} \varepsilon_j x_j\right\|^q \leq C^q \sum_{j=1}^{n} \|x_j\|^q ,$$

where $x_1, \ldots, x_n \in X$ and $\varepsilon_1, \ldots, \varepsilon_n$ i.i.d. Rademacher random variables.

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Examples:

- all Banach spaces have type 1 ("trivial type")
- Hilbert spaces have type 2
- L^p, ℓ^p spaces have type p for $p \in [1, 2]$

Pisier-like inequality on the symmetric group

Theorem (G., van Handel 2024+)

Let S_n denote the symmetric group with generator set

$$S = \{(ij) : i \neq j,\ i,j \in [n]\}.$$

If $(X, \|\cdot\|)$ is a Banach space of type $p \in [1,2]$ and $f : (S_n, S) \to (X, \|\cdot\|)$, then for $n \geq 2$,

$$\mathbb{E}_\mu \|f - \mathbb{E}_\mu f\| \lesssim \left(\frac{\log n}{n}\right)^{\frac{1}{p}} \left(\sum_{\substack{i,j=1,\\ i<j}}^{n} \mathbb{E}\|D_{ij} f(X_0)\|^p\right)^{\frac{1}{p}}.$$

Corollary: unbounded distortion in bilipschitz embedding of (S_n, S) into $(X, \|\cdot\|)$

We call $f : (M, d) \to (X, \|\cdot\|)$ a bilipschitz embedding with distortion $D \in \mathbb{R}$ if

$$d(x, y) \leq \|f(x) - f(y)\| \leq Dd(x, y)$$

for all $x, y \in M$.

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Corollary (G., van Handel 2024+)

For any $f : (S_n, S) \to (X, \|\cdot\|)$, a *bilipschitz embedding* with distortion D with $(X, \|\cdot\|)$ of type $p \in [1, 2]$,

$$D \gtrsim n^{1-\frac{1}{p}} \left(\frac{1}{\log n}\right)^{\frac{1}{p}}.$$

Remarks on the proof

Main techniques

How to "isolate" the $\delta_s(t)$s:

- decoupling and symmetrization arguments

Obtaining bounds on moments of the $\delta_s(t)$s:

- Small t: Bakry-Émery curvature and Gamma calculus
- Large t: Markov chain mixing

References

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